

Convergence of Time-Dependent Turing Structures to a Stationary Solution

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Abstract. Stability of stationary solutions of parabolic equations is conventionally studied by linear stability analysis, Lyapunov functions or lower and upper functions. We discuss here another approach based on differential inequalities written for the L^2 norm of the solution. This method is appropriate for the equations with time dependent coefficients. It yields new results and is applicable when the usual linearization method is not applicable.

Key words: parabolic systems, stationary solutions, stability, differential inequalities

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1. Formulation of the problem

Large-time behavior of solutions to differential equations has been discussed in many publications, see, for example, [2], [5], [13], [15]. First, one has to establish the global existence of the solution. This is done in most cases by establishing an a priori estimate which implies boundedness of the solutions for all times. The usual approach to Lyapunov stability of solutions is to linearize the problem and prove that the spectrum of the linearized operator lies strictly in the left half-plane of the complex plane.

In recent papers [4], [10], [11], [12], a novel approach to the stability and long-time behavior of solutions to abstract differential equations is developed. This approach is applied here to nonlinear systems of interest in biology.

Consider the semilinear parabolic system of equations

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$$\frac{\partial u}{\partial t} = D(t)\Delta u + F(u, x, t) \quad (1.1)$$

in a bounded domain $\Omega \subset \mathbb{R}^M$ with a sufficiently smooth boundary and with the homogeneous Dirichlet or Neumann boundary condition

$$u|_{\partial\Omega} = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad (1.2)$$

and the initial condition

$$u(x, 0) = u_0(x). \quad (1.3)$$

Here $u = (u_1, \dots, u_n)$, $F = (F_1, \dots, F_n)$, D is a diagonal matrix with positive diagonal elements $d_i = d_i(t)$, which can depend on t , and

$$F(0, x, t) = 0, \quad \forall x \in \Omega, t \geq 0. \quad (1.4)$$

The vector-function F is assumed to satisfy the estimates

$$\sup_{u, x \in \mathbb{R}^M, t \geq 0} |F(u, x, t)| \leq M_1, \quad (1.5)$$

$$|F(v, x, t) - F(w, x, s)| \leq c_F(|t - s| + |v - w|), \quad (1.6)$$

where $c_F > 0$ is a constant independent of v, w, x, t , and F is continuous with respect to x . Under these conditions, $u = 0$ is a stationary solution of problem (1.1), (1.2). In the examples considered below we assume that $M \leq 3$.

Consider the operator linearized about this solution:

$$L_t v = D\Delta v + F'_u(0, x, t)v$$

acting in the Hilbert space $L^2(\Omega)$ with the domain

$$\mathcal{D} = \{u \in H^2(\Omega), u|_{\partial\Omega} = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}.$$

Here t is considered as a parameter.

Suppose that the spectrum of the operator L_t is located in the half-plane $\text{Re } \lambda \leq \sigma(t)$. If

$$\sigma(t) \leq \sigma_0 < 0, \quad t \geq 0,$$

then solution of problem (1.1)-(1.3) converges to the stationary solution $u = 0$. This means that the stationary (equilibrium) solution is exponentially stable, i.e., the solutions with sufficiently small initial data converge to the stationary solution $u = 0$ at an exponential rate.

A proof of this assertion is well known in the case of the abstract evolution problem of the type

$$\dot{u} = Au + B(t, u), \quad u(0) = u_0, \quad (1.7)$$

where A is a linear bounded operator in a Banach space, with the spectrum that lies in the half-plane $\operatorname{Re} z \leq \sigma_0 < 0$, and $B(t, u)$ is a nonlinear operator satisfying the assumption

$$\|B(t, u)\| \leq c_0(t)\|u\|^p, \quad p > 1,$$

where $c_0(t)$ satisfies a suitable smallness assumption (see, e.g., [2]).

In [2], Theorem I.4.1, the following result is proved: if A is a bounded operator in a Banach space, then there exists the limit

$$\kappa := \lim_{t \rightarrow \infty} \frac{\ln \|e^{At}\|}{t} = \max\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the spectrum of A . Therefore, if all the solutions to the Cauchy problem

$$\dot{u} = Au, \quad u(0) = u_0,$$

decay exponentially fast to zero, then the spectrum of A lies in the half-plane $\operatorname{Re} z \leq -\kappa$, $\kappa > 0$, and vice versa. One should have in mind that if A is a bounded linear operator in a Hilbert space H , such that $\operatorname{Re} A \leq -\kappa$, $\kappa > 0$, i.e., $\operatorname{Re}(Au, u) \leq -\kappa\|u\|^2$, then the spectrum of A lies in the half-plane $\operatorname{Re} z \leq -\kappa$, but the converse of this statement is false if $\dim H > 1$: even in two-dimensional Hilbert space one can give an example of A with the spectrum lying in the half-plane $\operatorname{Re} z \leq -\kappa < 0$, for which the inequality $\operatorname{Re}(Au, u) \leq -\kappa\|u\|^2$ does not hold. For instance, let

$$A = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix},$$

The spectrum of this A consists of negative eigenvalue $\lambda = -1$. The quadratic form for real-valued u_1 and u_2 is $\operatorname{Re}(Au, u) = -u_1^2 - u_2^2 + 3u_1u_2$. This quadratic form is *not* negative-definite.

If the spectrum of A does not lie strictly in the left half-plane of the complex plane, or $\sigma(t) \rightarrow 0$ as $t \rightarrow \infty$, then the assertion about exponential rate of convergence to zero of the solutions to the Cauchy problem (1.7) is not valid, in general, and the Lyapunov stability problem cannot be solved by a study of the linearized problem.

In this work we study this, more difficult, case, and use a new technical tool for such a study, see Lemma 2.1. Let us emphasize that $\sigma(t)$ will not necessarily be assumed negative in this paper (see [12]).

2. Convergence of solutions

In what follows we assume that $F(u, x, t)$ satisfies assumptions made in Section 1, see (1.5) and (1.6). The initial and the boundary conditions satisfy the compatibility conditions, $u_0(x) = 0$ on $\partial\Omega$ for the Dirichlet and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ for the Neumann boundary condition. Under these (and some additional) conditions (see, e.g., [6]) there exists a classical solution of problem (1.1)-(1.3).

Let us assume that

$$F(u, x, t) = A(x, t)u + B(u, x, t),$$

where

$$\operatorname{Re} (A(x, t)u, u) \leq -\gamma(t)|u|^2, \quad \forall x \in \Omega, \quad t \geq 0, \quad (2.1)$$

and

$$|B(u, x, t)| \leq c_0(t)|u|^p, \quad \forall x \in \Omega, \quad t \geq 0, \quad p > 1. \quad (2.2)$$

Here (\cdot, \cdot) denotes the inner product in \mathbb{R}^3 , and $|u|^2 = \sum_{j=1}^n |u_j|^2$. We assume that the diagonal elements $d_i(t)$ of the matrix $D(t)$ of the diffusion coefficients satisfy the estimates

$$d_i(t) \geq d(t), \quad i = 1, \dots, n, \quad t \geq 0, \quad (2.3)$$

where $d(t)$ is a positive function. The assumptions about the behavior of $d(t)$ for large t will be specified below, in the formulation of Theorems 3.1-3.3.

Let $g(t) := \|u(\cdot, t)\|$, where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. We will also use the space $L^\infty(\Omega)$ with the norm $\|\cdot\|_\infty$, and the usual Sobolev space $H^2(\Omega)$ with the norm $\|\cdot\|_{H^2(\Omega)}$.

Multiplying equation (1.1) by u and integrating, we obtain, taking into account (2.1)-(2.3):

$$g\dot{g} \leq -d(t)\|\nabla u\|^2 - \gamma(t)g^2 + c_0(t) \int_{\Omega} |u|^{p+1} dx. \quad (2.4)$$

In the case of the Dirichlet boundary condition, we use the Poincaré inequality

$$c(\Omega) \int_{\Omega} |u|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx, \quad (2.5)$$

where $c(\Omega)$ is a positive constant which depends on the domain. The optimal (maximal possible) value of $c(\Omega)$ is equal to the first eigenvalue λ_1 of the Dirichlet Laplacian in Ω .

In the case of the Neumann boundary condition, we put $c(\Omega) = 0$.

Using the following multiplicative inequality (see, e.g., [1], p.193):

$$\|u\|_\infty \leq c\|u\|_{L^2(\Omega)}^{1/4} \|u\|_{H^2(\Omega)}^{3/4},$$

where the constant $c > 0$ is independent of u , we obtain

$$\int_{\Omega} |u|^{p+1} dx \leq g^2 \|u\|_\infty^{p-1} \leq c^{p-1} \|u\|_{H^2(\Omega)}^{3(p-1)/4} g^{(p+7)/4}.$$

From this inequality, (2.4) and (2.5) we obtain

$$\dot{g} \leq -(d(t)c(\Omega) + \gamma(t))g + c_0(t)c^{p-1} \|u\|_{H^2(\Omega)}^{3(p-1)/4} g^{(p+3)/4}. \quad (2.6)$$

It is known that under our assumptions the H^2 norm of the solution is bounded (see [3], Theorem 16.1, p.170, and Section 4).

Define

$$\sigma(t) := d(t)c(\Omega) + \gamma(t), \quad \alpha(t) := c_0(t)c^{p-1} \|u\|_{H^2(\Omega)}^{3(p-1)/4}, \quad q := \frac{p+3}{4}. \quad (2.7)$$

Then (2.6) can be written as

$$\dot{g} \leq -\sigma(t)g + \alpha(t)g^q, \quad g(t) \geq 0, \quad (2.8)$$

where $q > 1$ because $p > 1$.

Assume that $\sigma(t)$ and $\alpha(t) \geq 0$ are continuous functions defined on $[0, \infty)$.

We will use in Section 3 the following basic result from [10], where more general results are obtained (see also [12]):

Lemma 2.1. *If there exists a function $\mu(t) > 0$, defined on $[0, \infty)$, such that*

$$\alpha(t) \leq \mu^{q-1}(t) \left(\sigma(t) - \frac{\dot{\mu}(t)}{\mu(t)} \right), \quad t \geq 0 \quad (2.9)$$

and

$$\mu(0)g(0) \leq 1, \quad (2.10)$$

then $g(t)$ exists for all $t \geq 0$, and

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0. \quad (2.11)$$

Note that if $\lim_{t \rightarrow \infty} \mu(t) = \infty$, then estimate (2.11) implies that $\lim_{t \rightarrow \infty} g(t) = 0$. The function $\sigma(t)$ in lemma 2.1 is not necessarily positive.

3. Applications

A relatively general class of abstract differential equations for which our method is applicable is described by the equations of the form

$$\dot{u} = A(t)u + G(t, u) + f(t), \quad u(0) = u_0,$$

where A is a linear operator in a Hilbert space H , G is a nonlinear operator in H , and f is a given function with values in H . The following assumptions allow one to use our approach: $\operatorname{Re}(A(t)u, u) \leq -\gamma(t)\|u\|^2$, $\|G(t, u)\| \leq a(t, g)$, $g := \|u(t)\|$, $\|f(t)\| \leq \beta(t)$, where the functions $\gamma, a(t, g)$ and β satisfy some assumptions that are detailed in [4], [10].

In this section we will apply the results obtained above to reaction-diffusion system (1.1) with time dependent coefficients. In particular, in the case where the diffusion coefficients converge to zero and conventional results on stability of stationary solutions are not applicable.

3.1. Convergence with various rates

Exponential rate of convergence.

In order to make clear our method for a study of the large-time behavior of the solution to problem (1.1)-(1.3), let us consider first a single equation and the Dirichlet boundary condition.

Specifically, consider the following example:

$$A(x, t) \equiv a_0 > 0, \quad D(t) \equiv d_0 > 0, \quad c_0(t) \equiv c_0,$$

where a_0, d_0 and c_0 are some constants, and $p = 2$ in (2.2). Then $\gamma(t) = -a_0$, and

$$\sigma = d_0 c(\Omega) - a_0, \quad q = \frac{5}{4}.$$

If $\sigma > 0$, that is, if

$$\frac{a_0}{d_0} < c(\Omega), \tag{3.1}$$

then we choose

$$\mu(t) = \mu_0 e^{\nu t},$$

where μ_0 and ν are positive constants.

Let us formulate sufficient conditions for assumptions (2.9) and (2.10) to be satisfied. If these assumptions are satisfied, then inequality (2.11) yields an exponential rate of decay of the function $g(t)$, and, therefore, of the solution $u(t)$ to zero.

This assertion can be explained in terms of the exponential stability in the sense of Lyapunov of the solution $u = 0$ to the problem (1.1)-(1.3). Namely, consider the problem, linearized about the zero solution. The principal eigenvalue of the linearized problem

$$d_0 \Delta u + a_0 u = \lambda u, \quad u|_{\partial\Omega} = 0$$

becomes negative if

$$\frac{a_0}{d_0} < c(\Omega), \tag{3.2}$$

where the constant $c(\Omega)$ is from the Poincaré inequality (2.5). This is condition (3.1).

To satisfy assumption (2.10) one may choose

$$\mu_0 = g(0)^{-1}.$$

One may assume that $g(0) \neq 0$, because otherwise the solution is zero by the uniqueness theorem that holds under our assumptions.

To satisfy assumption (2.9) it is sufficient to assume that

$$C c_0(t) \leq g(0)^{-\frac{1}{4}} e^{\frac{1}{4}\nu t} [\sigma(t) - \nu], \tag{3.3}$$

where we took into account that $q-1 = \frac{1}{4}$ and denoted by C the constant $c^{p-1} \|u\|_{H^2(\Omega)}^{3/4}$. In Section 4 it is proved that the norm $\|u\|_{H^2(\Omega)}$ can be estimated from above by a constant independent of u . If one chooses $\nu = 0.5\sigma = (d_0\lambda_1 - a_0)/2$, then inequality (3.3) holds provided that

$$c_0(t) \leq 0.5g(0)^{-\frac{1}{4}} C^{-1} e^{\frac{1}{8}\sigma t} \sigma. \quad (3.4)$$

Thus, condition (3.4) is sufficient for the assumption (2.9) to be satisfied. Condition (3.4) holds for any fixed $g(0)$ if $c_0(t)$ is sufficiently small. It holds for a fixed $c_0(t)$ if $g(0)$ is sufficiently small.

We have proved the following result.

Theorem 3.1. *Let the function $\sigma(t)$, defined in (2.7), satisfy the inequality*

$$\sigma(t) \geq \sigma_0 > 0, \quad \forall t \geq 0,$$

where σ_0 is a constant. Choose $\mu(t) = \mu_0 \exp(\nu t)$, $\mu_0 = g(0)^{-1}$, $\nu = 0.5\sigma_0$. If condition (3.4) holds, then the L^2 norm of the solution $u(x, t)$ to problem (1.1)-(1.3) with the Dirichlet boundary condition satisfies the estimate

$$\|u(\cdot, t)\| \leq g(0)e^{-0.5\sigma_0 t}, \quad \forall t \geq 0. \quad (3.5)$$

The conclusion of this theorem follows from Lemma 2.1, see estimate (2.11).

The method for estimating the large time behavior of solutions to evolution problems, that was used in the proof of Theorem 3.1 is easy to apply in many problems.

The assumptions of Theorem 3.1 do not explicitly require that the spectrum of the linearized problem lies in the open left half-plane of the complex plane. However, the exponential rate of decay of the solution suggests that this is the case (see [2], p.42, p.51).

Convergence at a power rate.

Let us consider problem (1.1)-(1.3) with the Dirichlet boundary condition. Let us assume that

$$d(t) = \frac{d_0}{t+1}, \quad \gamma(t) = -\frac{\gamma_0}{(t+1)^k}, \quad \mu(t) = \mu_0(t+1)^m, \quad (3.6)$$

where $d(t)$ is the lower bound of the diffusion coefficients, see (2.3), d_0, γ_0 , and μ_0 are some positive constants, $k \geq 1$ is a constant. Then inequality (2.9) takes the form:

$$\alpha(t) \leq \mu_0^{q-1}(t+1)^{m(q-1)} \left(c(\Omega) \frac{d_0}{t+1} - \frac{\gamma_0}{(t+1)^k} - \frac{m}{t+1} \right), \quad t \geq 0 \quad (3.7)$$

Let us assume that

$$c(\Omega)d_0 > \gamma_0 + m. \quad (3.8)$$

If the above inequality holds, then the right-hand side of (3.7) is positive. This inequality gives a condition on the function $c_0(t)$, defined in equation (2.2) and used in the definition of $\alpha(t)$ in (2.7).

If $m(q-1) < 1$, then condition (3.7) implies that $c_0(t)$ should converge to zero as $t \rightarrow \infty$, if $m(q-1) > 1$, then it $c_0(t)$ may grow, as t grows, and still inequality (3.7) may be satisfied.

To satisfy assumption (2.10) one may choose

$$\mu_0 = g(0)^{-1}. \quad (3.9)$$

If (3.7), (3.8), and (3.9) hold, then one may apply estimate (2.11) and obtain the following theorem.

Theorem 3.2. *If conditions (3.6)- (3.9) are satisfied, then the L^2 norm of the solution $u(x, t)$ of problem (1.1)-(1.3) with the Dirichlet boundary condition admits the estimate*

$$\|u(\cdot, t)\| \leq g(0) (t+1)^{-m}, \quad \forall t \geq 0.$$

Boundedness of the solution.

Consider the case when global asymptotic stability of the stationary solution may not hold. We wish to obtain an estimate of the solution of the evolution problem, which yields stability in the sense of Lyapunov. We will illustrate the method in the case of Neumann boundary condition.

If the Neumann boundary condition holds, then, in contrast with the Dirichlet boundary condition, one has $c(\Omega) = 0$ in equation (2.7) and inequality (2.5), so one gets $\sigma(t) \equiv \gamma(t)$.

Let

$$\gamma(t) = \frac{\gamma_0}{1+t}, \quad \mu(t) = \mu_0(1+t)^m,$$

where γ_0 and μ_0 are some positive constants. If $\gamma_0 > m$ and $c_0(t)$ is such that inequalities (2.9) and (2.10) hold, i.e.,

$$\mu_0 = g(0)^{-1},$$

and

$$Cc_0(t) \leq \mu_0^{q-1}(1+t)^{m(q-1)} \frac{\gamma_0 - m}{1+t},$$

then inequality (2.11) yields convergence at the rate $O((1+t)^{-m})$. In this example $\gamma(t)$ is positive.

We can consider the case when $\gamma(t)$ is *negative*, but then $\mu(t)$ has to be a decreasing function. For instance, assume that

$$\gamma(t) = -\frac{\gamma_0}{(t+1)^k}, \quad \mu(t) = \mu_0 + \mu_1(t+1)^{-\nu}, \quad (3.10)$$

where the constants $\gamma_0, \mu_0, \mu_1 > 0$ and $\nu > 0$. In this case, (2.11) yields boundedness of the solution for all $t \geq 0$, but the solution does not converge to zero.

Inequality (2.9) takes the form:

$$\alpha(t) \leq (\mu_0 + \mu_1(t+1)^{-\nu})^{q-1} \left(\frac{\nu\mu_1(1+t)^{-\nu-1}}{\mu_0 + \mu_1(t+1)^{-\nu}} - \frac{\gamma_0}{(t+1)^k} \right). \quad (3.11)$$

This inequality holds if, for example,

$$\nu + 1 \leq k,$$

and, with $\alpha(t) = Cc_0(t)$, the following inequality holds:

$$C(1+t)^{\nu+1}c_0(t) \leq \mu_0^{q-1} \left(\frac{\nu\mu_1}{\mu_0} - \gamma_0 \right). \quad (3.12)$$

If (3.12) holds, and

$$\mu(0) = \mu_0 + \mu_1 = [g(0)]^{-1}, \quad (3.13)$$

then inequality (2.11) yields the following Theorem.

Theorem 3.3. *If conditions (3.10)-(3.12) hold, and $\nu + 1 \leq k$, then the L^2 norm of the solution $u(x, t)$ of problem (1.1)-(1.3) with the Neumann boundary condition satisfies the estimate*

$$\|u(\cdot, t)\| \leq [\mu(t)]^{-1} \leq [\mu_0]^{-1} \quad \forall t \geq 0.$$

3.2. Time-dependent Turing structures

Consider a reaction-diffusion system

$$\frac{\partial u}{\partial t} = d_1(t) \frac{\partial^2 u}{\partial x^2} + F(u, v, t), \quad (3.14)$$

$$\frac{\partial v}{\partial t} = d_2(t) \frac{\partial^2 v}{\partial x^2} + G(u, v, t) \quad (3.15)$$

in the interval $0 < x < L$ with the boundary conditions

$$u(0) = u(L) = 0, \quad v(0) = v(L) = 0. \quad (3.16)$$

Reaction-diffusion systems describe various applied problems, for example, biological problems. These systems are often considered in the case when the coefficients and the nonlinearities do not depend explicitly on time. We introduce time dependence in order to describe variations of the environment (e.g., climate factors), or to control the system behavior. For instance, if u and v are some concentrations, then the coefficients of mass diffusion and the reaction rates can depend on the temperature that can change in time due to some external conditions, or the temperature can serve as a control parameter.

Suppose that $F(0, 0, t) = G(0, 0, t) = 0$ for all $t \geq 0$, that is $u = v = 0$ is a stationary solution of problem (3.14)-(3.16). This zero solution is also a stationary solution to the ODE system

$$\frac{du}{dt} = F(u, v, t), \quad (3.17)$$

$$\frac{dv}{dt} = G(u, v, t). \quad (3.18)$$

To simplify calculations, let us assume that

$$F(u, v, t) = \phi(t) F^0(u, v), \quad G(u, v, t) = \phi(t) G^0(u, v), \quad d_i(t) = \phi(t) d_i^0, \quad i = 1, 2.$$

Consider first the case where $\phi(t) \equiv 1$. Let us choose parameters in such a way that $u = v = 0$ is a stable solution of system (3.17)-(3.18) but it is unstable as a solution of problem (3.14)-(3.16). In this case, another solution, which is not homogeneous in space, can appear. This is so-called *Turing structure*, that is often discussed in relation with numerous biological applications (see, e.g., [14], [7]-[9]). The Turing structure provides one of the possible mechanisms of pattern formation in biology.

We assume that the solution $u = v = 0$ of system (3.17), (3.18) is stable, and that the eigenvalues of the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

have negative real parts. Here

$$a = F_u^0(0, 0), \quad b = F_v^0(0, 0), \quad c = G_u^0(0, 0), \quad d = G_v^0(0, 0).$$

In order to study stability of the zero solution as a stationary solution of problem (3.14)-(3.16), consider the linearized system

$$\frac{\partial u}{\partial t} = d_1^0 \frac{\partial^2 u}{\partial x^2} + au + bv, \quad (3.19)$$

$$\frac{\partial v}{\partial t} = d_2^0 \frac{\partial^2 v}{\partial x^2} + cu + dv \quad (3.20)$$

with the boundary conditions (3.16).

If one looks for the solutions of this system in the form

$$u = p_1 \sin(kx) e^{\lambda t}, \quad v = p_2 \sin(kx) e^{\lambda t},$$

then one obtains the following eigenvalue problem:

$$M(k)p = \lambda p,$$

where

$$M(k) = \begin{pmatrix} a - d_1^0 k^2 & b \\ c & d - d_2^0 k^2 \end{pmatrix},$$

$p = (p_1, p_2)$, $M(0) = M$. Denote its eigenvalues by $\lambda_i(k)$, $i = 1, 2$. The assumption $\operatorname{Re} \lambda_i(0) < 0$, $i = 1, 2$ implies:

$$a + d < 0, \quad ad - bc > 0. \quad (3.21)$$

Furthermore,

$$\det M(k) = ad - bc - (ad_2^0 + dd_1^0)k^2 + d_1^0 d_2^0 k^4, \quad \text{Tr } M(k) = a + d - (d_1^0 + d_2^0)k^2.$$

If $\det M(k) = 0$, then one eigenvalue of this matrix is negative and another one equals zero. Hence, system (3.19), (3.20) linearized about the solution $u = v = 0$ has a zero eigenvalue. If, under a change of parameter, this eigenvalue crosses the origin, then a spatially inhomogeneous solution can bifurcate from it.

Thus, equality $\det M(k) = 0$ determines the stability boundary of the solution $u = v = 0$ and the condition of bifurcation of a spatially inhomogeneous solution.

Let us verify that equality $\det M(k) = 0$ is compatible with inequalities (3.21). If $a < 0$ and $d < 0$, then $\det M(k) > 0$. In order to have $\det M(k) = 0$, let us assume that one of the coefficients a or d is positive, but the sum $a + d$ is negative. Let us assume, for instance, that $a > 0$. The constants b, c, d can be chosen in such a way that inequalities (3.21) are satisfied. For some a, b, c, d, k, d_2^0 fixed, we can increase d_1^0 so that the determinant of the matrix $M(k)$ becomes zero.

Thus, if $\phi(t) \equiv 1$, then $u = v = 0$ can be a stable solution of system (3.17), (3.18), but unstable as a solution of problem (3.14)-(3.16). In this case, a stationary spatial structure can emerge and the solution of the evolution problem can converge to it.

If $\phi(t) \not\equiv 1$, then the previous considerations do not allow us to conclude whether the solution of problem (3.14)-(3.16) with a given initial condition converges to a trivial solution or to a spatially inhomogeneous solution.

Let us use the method developed in Section 2 in order to study the behavior of solution of the time dependent reaction-diffusion system. We have

$$\sigma(t) = \phi(t)(d_0 c(\Omega) - \gamma_0),$$

where

$$d_0 = \min(d_1^0, d_2^0),$$

$$au_1^2 + (b+c)u_1u_2 + du_2^2 \leq \left(a + \frac{1}{2}(b+c)\right)u_1^2 + \left(d + \frac{1}{2}(b+c)\right)u_2^2 \leq \gamma_0(u_1^2 + u_2^2),$$

$$\gamma_0 = \max\left(a + \frac{1}{2}(b+c), d + \frac{1}{2}(b+c)\right).$$

We obtain the following result.

Theorem 3.4. 1. Assume that $d_0 c(\Omega) > \gamma_0$,

$$\phi(t) = \frac{\phi_0}{t+1}, \quad \mu(t) = \mu_0(t+1)^m, \quad \mu_0^{-1} = g(0).$$

If $\phi_0(d_0 c(\Omega) - \gamma_0) > m$, and $c_0(t)$ (see (2.2)) and $\alpha(t)$ (see (2.7)) are such that condition (2.9) is satisfied, then

$$\|u(\cdot, t)\| \leq g(0)(t+1)^{-m}, \quad t \geq 0.$$

2. Let $d_0 c(\Omega) < \gamma_0$,

$$\mu(t) = \mu_0 + \frac{\mu_1}{(t+1)^k}, \quad (\mu_0 + \mu_1)^{-1} = g(0),$$

where μ_0, μ_1 and k are some positive constants.

If $c_0(t)$ and $\alpha(t)$ satisfy condition (2.9), then

$$\|u(\cdot, t)\| \leq \mu_0^{-1} \quad \forall t \geq 0.$$

The conclusion of this theorem follows from Lemma 2.1. The first part of the theorem gives a sufficient condition of convergence to the trivial solution. If this condition is not satisfied, then the solution can possibly converge to a spatially inhomogeneous solution. In this case, the second part of the theorem gives an estimate of the solution.

4. An estimate of the solution

Lemma 4.1. Suppose that for some positive constant M_1 the following estimate holds:

$$|F(u, x, t)| \leq M_1, \quad \forall u \in \mathbb{R}^n, \quad x \in \Omega, \quad t \geq 0. \quad (4.1)$$

Then solution of problem (1.1)-(1.3) with the Dirichlet boundary condition satisfies the estimate

$$\|u\|_{H^2(\Omega)} \leq M_2, \quad t \geq 0. \quad (4.2)$$

Proof. Each component u_i of the solution satisfies the problem

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + F_i(u, x, t), \quad (4.3)$$

$$u_i|_{\partial\Omega} = 0, \quad u_i(x, 0) = u_i^0(x). \quad (4.4)$$

We first obtain an estimate of the solution in the uniform norm. Let

$$v(x) = -a|x|^2 + b,$$

where

$$-2nad_i + M_1 \leq 0.$$

The ball with the radius $R = \sqrt{b/a}$ contains the domain Ω , and

$$u_i^0(x) \leq v(x), \quad x \in \Omega.$$

Such constants a and b can be chosen for any M_1 and any initial condition.

Then $v(x)$ is an upper solution of equation (4.3) and

$$u_i(x, t) \leq v(x), \quad x \in \Omega, \quad t \geq 0.$$

The functions $u_i(x, t)$ are bounded from below as well. Thus, estimate (4.2) follows from the known estimate (see, e.g., [3])

$$\|u\|_{H^2(\Omega)} \leq K \left(\|F\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right),$$

where $K > 0$ is a constant independent on u .

□

Lemma 4.2. Suppose that

$$F_i(u, x, t) \leq 0, \quad \forall u_i \geq u_i^*, \quad x \in \Omega, \quad t \geq 0, \quad (4.5)$$

for some constants u_i^* , $1 \leq i \leq n$. Then solution of problem (1.1)-(1.3) with the Neumann boundary condition satisfies estimate (4.2).

Proof. It is sufficient to note that any constant greater than u_i^* is an upper solution of equation (4.3).

□

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